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Simplicity of Eigenvalues in the Anderson Model

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We give a transparent and intuitive proof that all eigenvalues of the Anderson model in the region of localization are simple.

The Anderson tight binding model is given by the random Hamiltonian $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbf{Z}^d)$, where $\Delta(x, y) = 1$ if $|x - y| = 1$ and zero otherwise, and the random potential $V_\omega = \{V_\omega(x), x \in \mathbf{Z}^d\}$ consists of independent identically distributed random variables whose common probability distribution μ has a bounded density ρ. It is known to exhibit exponential localization at either high disorder or low energy.[1,3,4]

We prove a general result about eigenvalues of the Anderson Hamiltonian with fast decaying eigenfunctions, from which we conclude that in the region of exponential localization all eigenvalues are simple. We call $\varphi \in \ell^2(\mathbb{Z}^d)$ fast decaying if it has β-decay for some $\beta > \frac{5d}{2}$, that is, $|\varphi(x)| \leq C_\varphi \langle x \rangle^{-\beta}$ for some $C_\varphi < \infty$, where $\langle x \rangle := \sqrt{1 + |x|^2}$.

Theorem. The Anderson Hamiltonian H_ω cannot have an eigenvalue with two linearly independent fast decaying eigenfunctions with probability one.

We have the immediate corollary:

Corollary. Let I be an interval of exponential localization for the Anderson Hamiltonian H_ω. Then, with probability one, every eigenvalue of H_ω in I is simple.

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This corollary was originally obtained by Simon[8] as a consequence of a stronger result: in intervals of localization the vectors δ_x, $x \in \mathbb{Z}^d$, are cyclic for H_ω with probability one. Jaksic and Last[6] have recently extended Simon's ideas to prove that the singular spectrum of H_ω is almost surely simple. Simon's cyclicity result cannot be extended to Anderson-type Hamiltonians in the continuum.

Our proof is quite transparent and intuitive, and provides a new insight on the simplicity of eigenvalues. If an eigenvalue E of H has two linearly independent fast decaying eigenfunctions, then the corresponding finite volume operator must have at least two eigenvalues very close to E for large volumes. On the other hand, the probability of two eigenvalues of the finite volume operator being close together is very small for large volumes by an estimate due to Minami.[7] Since these two facts are incompatible, the eigenvalue E can have at most one fast decaying eigenfunction.

This insight should also hold in the continuum. The only step in our proof that cannot presently be done in the continuum is the use of Minami's estimate,[7] which is currently known only for the Anderson model. (See Appendix A for the statement of Minami's inequality and an outline of its proof.) We expect this estimate to hold in the continuum in some form. When Minami's estimate is extended to the continuum, our proof will give the simplicity of eigenvalues also for continuous Anderson-type Hamiltonians.

While the simplicity of eigenvalues for Anderson-type Hamiltonians in the continuum is not presently known, they are known to have finite multiplicity in the region of complete localization (i.e., the region of applicability of the multiscale analysis). Combes and Hislop[2] proved it for Anderson-type Hamiltonians in the continuum with bounded density for the probability distribution of the strength of single site potential. Recently, Germinet and Klein[5] proved finite multiplicity for all eigenvalues in the region of complete localization without any extra hypotheses than the availability of the multiscale analysis; in particular, their result does not require the probability distribution of the strength of single site potential to have a density.

The proof of the theorem is based on two lemmas regarding the finite volume operators, the first one a deterministic result.

We let Λ_L be the open box centered at the origin with side of length $L > 0$, and write χ_L for its characteristic function. Given $H = -\Delta + V$, we let H_L be the operator H restricted to $\ell^2(\Lambda_L)$ with zero boundary conditions outside Λ_L. We identify $\ell^2(\Lambda_L)$ with $\chi_L \ell^2(\mathbb{Z}^d)$, in which case $H_L = \chi_L H \chi_L$. We write $H_L^\perp = (1 - \chi_L) H (1 - \chi_L)$, and $\Gamma_L = H - H_L - H_L^\perp = -\Delta + \Delta_L + \Delta_L^\perp$. By $C_{a,b,\ldots}$ we will always denote some finite constant depending only on $a, b, \ldots$. We write χ_J for the charateristic function of the set J.

Lemma 1. *Let E be an eigenvalue for $H = -\Delta + V$ with two linearly independent eigenfunctions with β-decay for some $\beta > \frac{d}{2}$. Then there exists $C = C_{d,\beta,\varphi_1,\varphi_2}$,* *where φ_1 and φ_2 are the two eigenfunctions, such that if we set $\varepsilon_L = CL^{-\beta+\frac{d}{2}}$ and $J_L = [E - \varepsilon_L, E + \varepsilon_L]$, we have tr $\chi_{J_L}(H_L) \geq 2$ for all sufficiently large L.*

Proof: Let $\varphi_i \in \ell^2(\mathbb{Z}^d)$, $i = 1, 2$, be orthonormal with β-decay such that $H\varphi_i = E\varphi_i$. Given $\varphi \in \ell^2(\mathbb{Z}^d)$ we set $\varphi_L = \chi_L \varphi$ and $\varphi_L^\perp = \varphi - \varphi_L$. We have

$$\|\varphi_{i,L}^\perp\| \leq \varepsilon_L \quad \text{and} \quad \|\varphi_{i,L}\| \geq \sqrt{1 - \varepsilon_L^2}, \quad i = 1, 2, \tag{1}$$

$$\left|\langle \varphi_{1,L}, \varphi_{2,L}\rangle\right| \leq \varepsilon_L^2, \tag{2}$$

$$\|(H_L - E)\varphi_{i,L}\| = \|\Gamma_L \varphi_{i,L}^\perp\| \leq C'_{d,\beta,\varphi_1,\varphi_2} L^{-\beta+\frac{d-1}{2}} \leq \varepsilon_L, \quad i = 1, 2, \tag{3}$$

for all large L (assumed from now on), with $\varepsilon_L = C_{d,\beta,\varphi_1,\varphi_2} L^{-\beta+\frac{d}{2}}$.

It follows that $\varphi_{1,L}$ and $\varphi_{2,L}$ are linearly independent, and hence their linear span V_L has dimension two. Moreover, we can check that

$$\|(H_L - E)\psi\| \leq 2\varepsilon_L \|\psi\| \quad \text{for all} \quad \psi \in V_L. \tag{4}$$

Now let $J_L = [E - 3\varepsilon_L, E + 3\varepsilon_L]$, and set $P_L = \chi_{J_L}(H_L)$, $Q_L = I - P_L$. Then for all $\psi \in V_L$ we have, using (4),

$$\begin{aligned} \|Q_L\psi\| &\leq \| (3\varepsilon_L)^{-1} \|(H_L - E)Q_L\psi\| = (3\varepsilon_L)^{-1} \|Q_L(H_L - E)\psi\| \\ &\leq (3\varepsilon_L)^{-1} \|(H_L - E)\psi\| \leq \frac{2}{3}\|\psi\|, \end{aligned} \tag{5}$$

and hence

$$\|P_L\psi\|^2 = \|\psi\|^2 - \|Q_L\psi\|^2 \geq \frac{5}{9}\|\psi\|^2. \tag{6}$$

Thus P_L is injective on V_L and we conclude that tr$P_L \geq \dim V_L = 2$.

Redefining the constant in the definition of ε_L we get the lemma. □

The second lemma is probabilistic; it says that the probability of two eigenvalues (perhaps equal) of the finite volume operator being close together is very small for large volumes. It depends crucially on the following beautiful estimate of Minami ([7], Lemma 2 and proof of Eq. (2.48)):

$$\mathbb{P}\left\{\mathrm{tr}\chi_J(H_{\omega,L}) \geq 2\right\} \leq \pi^2 \|\rho\|_\infty^2 |J|^2 L^{2d} \tag{7}$$

for all intervals J and length scales $L \geq 1$. Since Minami's estimate is the heart of our proof, we outline its proof in Appendix A.

Lemma 2. *Let H_ω be the Anderson Hamiltonian. If I is a bounded interval and $q > 2d$, let $\mathcal{E}_{L,I,q}$ denote the event that* tr$\chi_J(H_{\omega,L}) \leq 1$ *for all subintervals $J \subset I$*

with length $|J| \le L^{-q}$. Then

$$\mathbb{P}\{\mathcal{E}_{L,I,q}\} \ge 1 - 8\pi^2\|\rho\|_\infty^2(|I|+1)L^{-q+2d}. \tag{8}$$

Proof: We can cover the interval I by $2\left(\left[\frac{L^q}{2}|I|\right]+1\right) \le L^q|I|+2$ intervals of length $2L^{-q}$, in such a way that any subinterval $J \subset I$ with length $|J| \le L^{-q}$ will be contained in one of these intervals. ($[x]$ denotes the largest integer $\le x$.) Since the complementary event, $\mathcal{E}_{L,I,q}^c$, occurs if there exists an interval $J \subset I$ with $|J| \le L^{-q}$ such that $\operatorname{tr}\chi_J(H_{\omega,L}) \ge 2$, its probability can be estimated, using (7), by

$$\mathbb{P}\{\mathcal{E}_{L,I,q}^c\} \le \pi^2\|\rho\|_\infty^2(L^q|I|+2)(2L^{-q})^2L^{2d} \le 8\pi^2\|\rho\|_\infty^2(|I|+1)L^{-q+2d}, \tag{9}$$

and hence (8) follows. □

Proof of Theorem: Let I be a bounded open interval, and set $L_k = 2^k$ for $k = 1, 2, \ldots$. It follows from Lemma 2, applying the Borel-Cantelli Lemma, that if $q > 2d$, then for P-a.e. ω there exists $k(q,\omega) < \infty$ such that the event $\mathcal{E}_{L_k,I,q}$ occurs for all $k \ge k(q,\omega)$. But if $E \in I$ is an eigenvalue for H_ω with two linearly independent eigenfunctions with β-decay for some $\beta > \frac{5d}{2}$, then Lemma 1 tells us that for all large k we have $\operatorname{tr}\chi_{J_k}(H_{\omega,L_k}) \ge 2$, where $J_k = J_{L_k}$ is a subinterval of I with $|J_k| \le CL_k^{-(\beta-\frac{d}{2})}$, which is not possible since if $\beta > \frac{5d}{2}$ there exists $q > 2d$ such that $\beta - \frac{d}{2} > q$. □

A. MINAMI'S ESTIMATE

In this appendix we state Minami's estimate (in two useful forms) and outline the steps in its proof.

Minami's estimate[7]: Let H_ω be the Anderson Hamiltonian. Then

$$\mathbb{P}\left\{\operatorname{tr}\chi_J(H_{\omega,L}) \ge 2\right\} \le \mathrm{E}\left\{\left\{\operatorname{tr}\chi_J(H_{\omega,L})\right\}^2 - \operatorname{tr}\chi_J(H_{\omega,L})\right\} \le \pi^2\|\rho\|_\infty^2|J|^2L^{2d} \tag{10}$$

for all intervals J and length scales $L \ge 1$.

Outline of the proof: Let $J = [E-\eta, E+\eta]$ be an interval, in which case

$$\chi_J(\lambda) \le 2\eta\,\Im\,(\lambda - (E+i\eta))^{-1} \quad \text{for all} \quad \lambda \in \mathrm{R}. \tag{11}$$

Thus, with $R_L(z) = (H_L - z)^{-1}$ and $G_L(z;x,y) = \langle\delta_x, R_L(z)\delta_y\rangle$, we have

$$\mathbb{P}\{\operatorname{tr}\chi_J(H_{\omega,L}) \ge 2\} \le \mathrm{E}\{(\operatorname{tr}\chi_J(H_{\omega,L}))^2 - \operatorname{tr}\chi_J(H_{\omega,L})\} \tag{12}$$

$$= \mathbb{E}\left\{\sum_{E_1,E_2\in\sigma(H_L);\, E_1\neq E_2} \chi_J(E_1)\chi_J(E_2)\right\} \tag{13}$$

$$\leq \mathbb{E}\left\{\sum_{E_1,E_2\in\sigma(H_L);\, E_1\neq E_2} \Im\frac{2\eta}{E_1-(E+i\eta)}\,\Im\frac{2\eta}{E_2-(E+i\eta)}\right\} \tag{14}$$

$$= (2\eta)^2\mathbb{E}\left\{\left(\mathrm{tr}\Im R_L(E+i\eta)\right)^2 - \mathrm{tr}\left\{(\Im R_L(E+i\eta))^2\right\}\right\} \tag{15}$$

$$= (2\eta)^2\sum_{x,y\in\Lambda_L}\mathbb{E}\left\{\det\begin{bmatrix}\Im G_L(E+i\eta;x,x) & \Im G_L(E+i\eta;x,y)\\ \Im G_L(E+i\eta;y,x) & \Im G_L(E+i\eta;y,y)\end{bmatrix}\right\} \tag{16}$$

$$\leq (2\eta)^2\pi^2\|\rho\|_\infty^2 L^{2d} = \pi^2\|\rho\|_\infty^2|J|^2L^{2d}, \tag{17}$$

where (14)–(16) is given in ([7], Eq. (2.64)) and (17) follows from ([7], Lemma 2). □

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